$\boxed{\text{Sec 5.1}}$ The Basics of Counting

Combinatorics, the study of arrangements of objects, is an important part of discrete mathematics. In this chapter, we will learn basic techniques of counting which has a lot of applications, e.g. how many different North American telephone numbers are possible? How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school? What is the chance of getting a full house in a poker hand?

The Product Rule Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example 1 The chairs of an auditorium are to be labeled with a letter followed by a positive number not exceeding 100. What is the largest number of chairs that can be labeled differently?

The product rule extends to the case a procedure can be broken down into a sequence of more than just two tasks. Suppose that the procedure is carried out by performing the tasks $T_1, T_2, \cdots, T_k$ in sequence. If each tasks T_i can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 n_2 \cdots n_k$ ways to carry out the procedure.

Example 2 How many different license plates are available if each plate contains a sequence of three letters followed by three digits?

Example 3 Let $A = \{1, 2, \cdots, m\}$ and $B = \{1, 2, \cdots, n\}$.

1. How many functions are there from A to B?

2. How many one-to-one functions are there from A to B?

Example 4 The telephone numbers in North America must consist of 10 digits, which are split into a three-digit area code, three-digit office code, and four-digit station code. According to the regulations of the *North American Numbering Plan*, each number should be of the form $NXX - NXX - XXXX$, where N denotes a digit that can take any of the values 2 through 9, and X denotes a digit that can take any of the values 0 through 9. How many different North American telephone numbers are possible?

We now introduce the sum rule.

The Sum Rule If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task.

Example 5 A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Many counting problems cannot be solved using just the product rule or just the sum rule. However, many complicated counting problems can be solved using *both* of these rules in combination.

Example 6 Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

$\boxed{\textbf{Sec 5.2}}$ The Pigeonhole Principle

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes, then we can say for sure that at least one of these 19 pigeonholes must have at least two pigeons in it. This illustrates a general principle called *Pigeonhole Principle.*

Pigeonhole Principle Let k be a positive integer. If $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of objects.

Example 1 A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

Example 2 In any group of 27 English words, there must be at least two that begin with the same letter.

Example 3 Assume that there were more than $100,000,000$ wage earners in the United States who earned less than $1,000,000$ dollars. Show that there are two who earned exactly the same amount of money, to the penny, last year.

Example 4 Let d be a positive integer. Show that among any group of $d + 1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

Example 5 Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Now we want to generalize Pigeonhole Principle. Suppose 21 objects are distributed into 10 boxes, then at least one box must have at least 3 objects. This is a special case of the following theorem:

The Generalized Pigeonhole Principle If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Proof We use the proof by contradiction. Suppose not, that is, each box contains at most ________ objects in it. Then the total number of objects is at most ________. Note that $\lceil \frac{N}{k} \rceil < \frac{N}{k} + 1$, hence we have

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

which gives a contradiction.

Example 6 Among 100 people there are at least $\lceil \frac{100}{12} \rceil = 9$ who were born in the same month.

Example 7 There are 38 different time periods during which classes at a college can be scheduled. If there are 677 different classes, how many different rooms will be needed?

$\boxed{\text{Sec 5.3}}$ Permutations and Combinations

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. We begin by considering an example.

Example 1 In how many ways can we select three students from a group of five students to stand in line for a picture. In how many ways can we arrange all five of these students in a line for a picture?

Definition A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

Example 2 Let $S = \{1, 2, 3, 4, 5\}$. The ordered arrangement $(4, 2, 1, 5, 3)$ is a permutation of S. $(3, 1, 4)$ is a 3-permutation of S.

Definition The number of r-permutations of a set with n elements is denoted by $P(n, r)$ or ${}_nP_r$.

We want to compute $P(n, r)$.

Example 3 Compute $P(5, 2)$ and $P(5, 3)$. What is $P(5, 5)$?

In general, we get

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)}_{r \text{ factors}} = \frac{n!}{(n-r)!}.$$

In particular,

$$P(n, n) = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!.$$

Example 4 How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Example 5 How many permutations of the letters $ABCDEFG$ contain the string ABC?

We now turn our attention to counting unordered selection of objects.

Example 6 How many different committees of two students can be formed from a group of four students?

Example 7 In how many ways can we choose a chair and a vice chair from a group of four students? How is this example different from the previous one?

Definition An r-combination of elements of a set is an unordered selection of r elements from the set.

Example 8 Let $S = \{1, 2, 3, 4, 5\}$. Then $\{1, 3, 4\}$ is a 3-combination of S.

Definition The number of r-combinations of a set with n elements is denoted by $C(n, r)$, ${}_nC_r$, or $\binom{n}{r}$.

Example 9 From **Example 6**, we see that $\binom{4}{2} =$____.

Example 10 Compute $\binom{5}{2}$. Compute $\binom{5}{3}$.

In general, we have the following theorem.

Theorem The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Sketch of Proof We explain with $\binom{5}{3}$. Let $S = \{1, 2, 3, 4, 5\}$ and consider all 3-permutations of S consisting of $\{1, 2, 4\}$. They are:

$$(1, 2, 4), (1, 4, 2), (2, 1, 4), (2, 4, 1), (4, 1, 2), \text{ and } (4, 2, 1).$$

On the other hand, there is only one 3-combination of S consisting of $\{1,2,4\}$; just $\{1,2,4\}$. Hence there are 6 times as many permutations as combinations consisting of $\{1,2,4\}$. Here 6 comes from the total number of different permutations of $\{1,2,4\}$, that is, $6 = 3!$. Since this is true for all 3-combinations of S, we conclude that

$$P(5,3) = 3! \times \binom{5}{3}.$$

In other words,

$$\binom{5}{3} = \frac{1}{3!} \times P(5,3) = \frac{1}{3!} \times \frac{5!}{(5-2)!} = \frac{5!}{3!(5-2)!}.$$

Example 11 Prove that $\binom{n}{r} = \binom{n}{n-r}$.

Example 12 How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Example 13 Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department in a college. How many ways are there to select a committee to develop a discrete mathematics course at the school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?